10/20/21

Calculus III

Last Time: Fubini's Thm

- If f is cts on $R = [a,b] \times [c,a]$,

then $\int_{y=c}^{d} \int_{x=a}^{b} f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA$

$= \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dy\,dx$

Ex: Compute $\iint_R \frac{x}{1+xy}\,dA$ on $R = [0,1] \times [0,1]$.

Sol 1: $\iint_R \frac{x}{1+xy}\,dA = \int_{x=0}^{1} \int_{y=0}^{1} \frac{x}{1+xy}\,dy\,dx$

Inner Integral: $\int_{y=0}^{1} \frac{x}{1+xy}\,dy$ $\quad u(y) = 1+xy$, $du = x\,dy$

$= \int_{y=0}^{1} \frac{1}{1+xy} \cdot x\,dy$

$= \int_{y=0}^{1} \frac{1}{u}\,du$

$= \ln|u| \Big|_{y=0}^{1}$

$= \ln|1+xy| \Big|_{y=0}^{1}$

$= \ln(1+x) - \ln(1+0)$

$= \ln(1+x)$

Outer Integral: $\int_{x=0}^{1} \ln(1+x)\,dx$

$= \left[x\ln(1+x) - \int \frac{x}{1+x}\,dx\right]_{x=0}^{1}$ $\quad u = \ln(1+x)$, $du = \frac{1}{1+x}\,dx$; $dv = dx$, $v = x$

$$= \left[x\ln(1+x) - \int \left(1 - \frac{1}{1+x}\right) dx \right]_{x=0}^{1}$$

$$= \left[x\ln(1+x) - (x - \ln(1+x)) \right]_{x=0}^{1}$$

$$= \left(\ln(2) - (1 - \ln(2))\right) - (0 - (0 - \ln(1)))$$

$$= 2\ln(2) - 1$$

Inner Integral: $\int_{x=0}^{1} \frac{x}{1+xy} dx$

↖ hard

In principle: could try $\begin{cases} u(x) = 1+xy \\ du = y\,dx \end{cases}$ ↘

$$= \int_{x=0}^{1} \frac{\left(\frac{u-1}{y}\right)}{u} \cdot \frac{1}{y} du \qquad x = \frac{u-1}{y}$$

$$= \frac{1}{y^2} \int_{x=0}^{1} \frac{u-1}{u} du$$

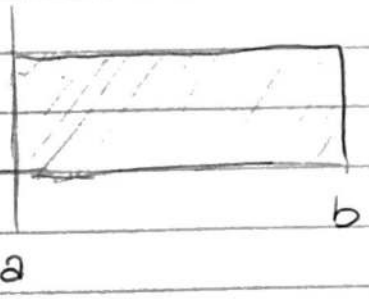
↑ contributes to disconnuity if $y=0$

ABANDON

Exercise: Compare $\iint_R y e^{-xy} dA$ on $R = [0,2] \times [0,3]$. Write out both possible orders of integration ...

Point: Sometimes one order is more computable than another order.

Defn: The average value of function $f(x,y)$ on region R is $\frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$



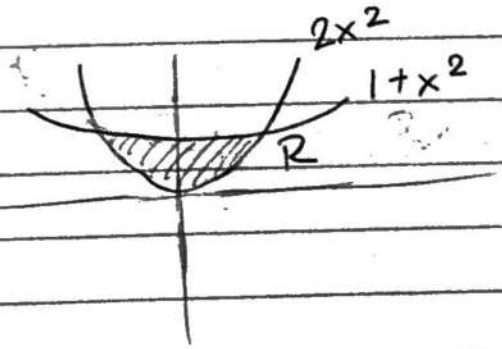
Goal: Integrate over more than just rectangles...

Ex: Compute the (net) volume of the solid bounded by $y = 2x^2$, $y = 1+x^2$, and $z = x + 2y$ and $z = 0$.

Net volume is $\iint_R ((x+2y) - 0)\,dA$

over $R = \{(x,y) : (x,y)$ between $y = 2x^2$ and $y = 1+x^2\}$

Now a picture of region in xy-plane:



For fixed x, we know $2x^2 \leq y \leq 1+x^2$.

To find x-bands solve $(2x^2 = 1+x^2$ iff $x = \pm 1)$

$$\therefore R = \left\{ (x,y) : -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2 \right\}$$

Thus, because our parameterization of R is ___, we can write our double integral as an iterated integral!

Sol: We just saw $R = \left\{ (x,y) : -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2 \right\}$

$$\therefore \iint_R (x+2y)\,dA = \int_{x=-1}^{1} \int_{y=2x^2}^{1+x^2} (x+2y)\,dy\,dx$$

$$= \int_{x=-1}^{1} \left[xy + y^2 \right]_{y=2x^2}^{1+x^2} dx$$

$$= \int_{x=-1}^{1} \Big[\left(x(1+x^2) + (1+x^2)^2 \right) + \left(x(2x^2) + (2x^2)^2 \right) \Big] dx$$

$$= \int_{x=-1}^{1} \left(x(1+x^2-2x^2) + \left((1+x^2)^2 - (2x^2)^2 \right) \right) dx$$

$$= \int_{x=-1}^{1} \left(x(1-x^2) + (1+x^2+2x^2)(1+x^2-2x^2) \right) dx$$

$$= \int_{x=-1}^{1} (1+x+3x^2)(1-x^2)\,dx$$

$$= \int_{x=-1}^{1} (1+x+2x^2-x^3-3x^4)\,dx$$

$$= \left[x + \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 - \frac{3}{5}x^5 \right]_{x=-1}^{1}$$

$$= \left(1 + \frac{1}{2} + \frac{2}{3} - \frac{1}{4} - \frac{3}{5}\right) - \left(-1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} - \frac{3}{5}\right)$$

$$= \frac{32}{15}$$

TAKEAWAY: If R is parameterized by something like:

$$R = \left\{(x,y) : c_1 \le x \le c_2 \;,\; g_1(x) \le y \le g_2(x)\right\},$$

then $$\iint_R f(x,y)\,dx = \int_{x=c_1}^{c_2} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Similarly, $R = \left\{(x,y) : c_1 \le y \le c_2 \;,\; g_1(y) \le x \le g_2(y)\right\}$

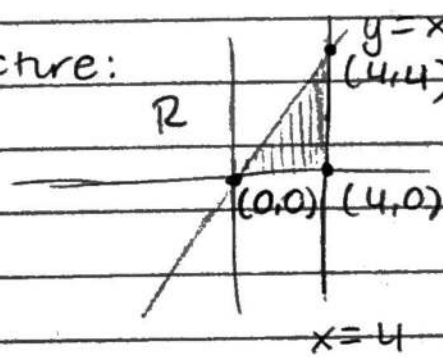
yields $$\iint_R f(x,y)\,dA = \int_{y=c_1}^{c_2} \int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

Ex: Compute $\iint_R y^2 e^{xy}\,dA$ for R bounded by

$y = x$, $y = 0$, $x = 4$.

$$R = \left\{(x,y) : 0 \le y \le 4,\; y \le x \le 4\right\}$$

Picture:



$$R = \left\{(x,y) : 0 \le x \le 4,\; 0 \le y \le x\right\}$$

Sol: $\iint_R y^2 e^{xy}\,dA = \int_{y=0}^{4}\int_{x=y}^{4} y^2 e^{xy}\,dx\,dy$ ← Easier

$=$

$\int_{x=0}^{4}\int_{y=0}^{x} y^2 e^{xy}\,dy\,dx$

$\iint_R y^2 e^{xy}\,dA = \int_{y=0}^{4}\int_{x=y}^{4} y^2 e^{xy}\,dx\,dy$

Inner: $\int_{x=y}^{4} y^2 e^{xy}\,dx = \int_{x=y}^{4} y e^{xy}\, y\,dx$

$= \int_{x=y}^{4} y e^{v}\,dv$ $\quad v = xy,\ dv = y\,dx$

$= y\left[e^{v}\right]_{x=y}^{v}$

$= y\left[e^{xy}\right]_{x=y}^{4}$

$= y(e^{4y} - e^{y^2})$

$= ye^{4y} - ye^{y^2}$

$\therefore \iint_R y^2 e^{xy} = \int_{y=0}^{4} (ye^{4y} - ye^{y^2})\,dy$

$= \int_{y=0}^{4} ye^{4y}\,dy - \int_{y=0}^{4} ye^{y^2}\,dy$

↑ Integration by parts

$v = y$, $dv = dy$, $dv = e^{4y}dv$, $v = \frac{1}{4}e^{4y}$

$= \left[\frac{1}{4}ye^{4y} - \int \frac{1}{4}e^{4y}dy - \frac{1}{2}\int e^{v}dv\right]_{y=0}^{4}$

$$= \left[\frac{1}{4} y e^{4y} - \frac{1}{16} e^{4y} - \frac{1}{2} e^{y^2} \right]_{y=0}^{4}$$

$$= \left(e^{16} - \frac{1}{16} e^{16} - \frac{1}{2} e^{16} \right)$$

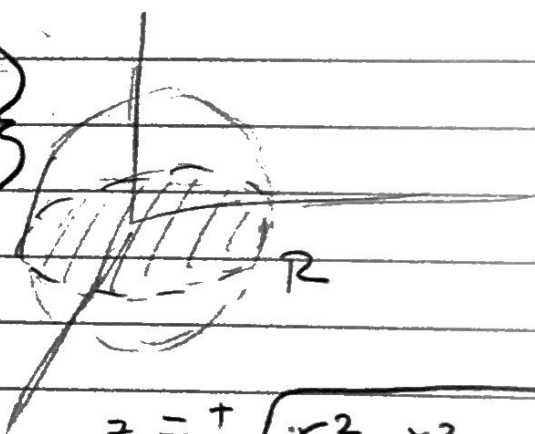
$$= \frac{1}{16}\left(1 - e^{16}\right) + \frac{1}{2} - \frac{1}{2} e^{16}$$

Motivating Question: What is the volume of the sphere?

Set up:

$$S = \{(x,y,z) : x^2+y^2+z^2 = r^2\}$$

$$R = \{(x,y) : -r \le x \le r, \; -\sqrt{r^2-x^2} \le y \le \sqrt{r^2-x^2}\}$$



We need to parameterize this region

$$z = \pm\sqrt{r^2-x^2-y^2}$$